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TN 50

# On the asymptotic behaviour of the solution of a differential-difference equation arising in number theory

by

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February 1968

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1. In this report we consider the asymptotic behaviour of the solution of a differential-difference equation which arose in a problem of number theory considered by the pure mathematics department.

Let $g(m)$ be the largest prime factor of a natural number m and let $G_\alpha(n)$ denote the number of integers $m \leq n$, for which $g(m) \leq m^\alpha$, where $\alpha$ is a real number. Then according to VAN DE LUNE and WATTEL [8] the frequency $G(\alpha)$ is defined as

$$G(\alpha) = \lim_{n\to\infty} G_\alpha(n)/n.$$

In their report the following differential-difference equation for $H(x) = G(1/x)$ is derived

$$xH'(x) = -H(x-1), \qquad -\infty < x < \infty. \tag{1.1}$$

The function $H(x)$ has to satisfy the condition

$$H(x) = 1, \qquad 0 \leq x \leq 1. \tag{1.2}$$

It follows from these equations that

$$H(x) = 0, \qquad x < 0. \tag{1.3}$$

In this report we shall investigate the asymptotic behaviour of $H(x)$ for large values of x.

2. We apply a Laplace transformation to (1.1) with respect to x and define

$$h(s) = \int_0^\infty e^{-sx} H(x)dx \tag{2.1}$$

for Re $s > s_0$.

The existence of $s_0$ follows from the boundedness of $H(x)$.

Integration by parts of the integral in

$$\int_{x-1}^{x} \{\xi H'(\xi) + H(\xi - 1)\}d\xi = 0$$

gives the recurrence relation

$$\int_{x-1}^{x} H(\xi)d\xi - xH(x) = \int_{x-2}^{x-1} H(\xi)d\xi - (x-1)\, H(x-1),$$

from which follows, since $H(x) < 0$ if $x < 0$, that

$$\int_{x-1}^{x} H(\xi)d\xi = x\, H(x).$$

Therefore, the assumption that there is a point $x_0 > 0$ for which $H(x_0) = 0$ leads to a contradiction, so that $H(x) > 0$ for $x > 0$.

From (1.1) follows that for $x > 1$ $H(x)$ is a monotonously decreasing function of $x$, so that $H(x) < 1$.

Hence

$$0 \leq H(x) \leq 1 \tag{2.2}$$

and a $s_0$ exists.

We remark that from its definition follows that $H(x)$ in the original number theory problem can be interpreted as a probability and therefore trivially satisfies (2.2).

With the aid of the transformation the differential-difference equation is transformed into the following simple ordinary differential equation

$$sh'(s) = (e^{-s} - 1)\, h(s), \qquad \mathrm{Re}\, s > s_0. \tag{2.3}$$

Since $H(x) \to 1$ as $x \to 0$ we obtain the boundary condition $sh(s) \to 1$ as $s \to \infty$.

The general solution of (2.3) is

$$h(s) = \frac{c}{s} \exp - E_1(s), \tag{2.4}$$

where

$$E_1(s) \overset{\mathrm{def}}{=} \int_s^\infty \frac{e^{-\sigma}}{\sigma}\, d\sigma.$$

The boundary condition gives at once $c = 1$.

From (2.4) follows that $h(s)$ is an entire function. Hence (2.1) is valid in the whole s-plane.

We note in passing that by substitution of $s = 0$ into (2.1) and

(2.4) the following curious relation is obtained

$$\int_0^\infty H(x)dx = \exp \gamma, \tag{2.5}$$

where $\gamma$ is Euler's constant.

Applying the inverse Laplace transformation to (2.1), we get

$$H(x) = \frac{1}{2\pi i} \int_L \exp(sx - E_1(s) - \ln s)ds, \tag{2.6}$$

where L denotes a vertical path in the s-plane.

The asymptotic behaviour of H(x) as $x \to \infty$ is determined by applying the saddle point method. Since

$$\Phi(s, x) = s - \frac{1}{x}(E_1(s) + \ln s) \tag{2.7}$$

is an entire function of s, there are no further singularities. The saddle points are found from $\partial\Phi/\partial s = 0$ which gives

$$\frac{1 - e^{-s}}{s} = x. \tag{2.8}$$

In order to facilitate the computations, we write (2.8) as

$$\frac{e^t - 1}{t} = x, \tag{2.9}$$

where $s = -t$. This equation can be brought in the following more symmetrical form by making the substitution

$$t = w - \frac{1}{x}.$$

This gives

$$we^{-w} = \frac{1}{x} e^{-\frac{1}{x}}, \tag{2.10}$$

which is of the form of an equation considered by WRIGHT [9]. There are two real roots and an infinity of complex roots. The trivial root $w = x^{-1}$ is of no importance to us but there exists a non-trivial one for which, however, no explicit analytical expression is available.

Hence there exists only one real saddle point.

It will be shown later on that only the real saddle point contributes to the asymptotic behaviour of $H(x)$.

The value $t_0$ of $t$ at the real saddle point can be easily approximated by means of an iterative method such as given in DE BRUIJN [3]. With $t = t_0$ we can write (2.8) in the form

$$t_0 = \ln x + \ln t_0 - \ln(1 - e^{-t_0}) \tag{2.11}$$

and proceed in the following way. From the rough estimate

$$\ln x < t_0 < 2 \ln x$$

obtained from (2.9) for $t = t_0$, we derive

$$\ln t_0 = O(\ln \ln x), \qquad x \to \infty,$$

which is substituted into the right-hand side of (2.11). This will give a better estimate for $\ln t_0$, viz.

$$\ln t_0 = \ln \ln x + O\left(\frac{\ln \ln x}{\ln x}\right), \qquad x \to \infty.$$

By making a further step in the iterative procedure we obtain the more precise result

$$\ln t_0 = \ln \ln x + \frac{\ln \ln x}{\ln x} - \tfrac{1}{2}\left(\frac{\ln \ln x}{\ln x}\right)^2 + O\left(\frac{\ln \ln x}{\ln^2 x}\right), \quad x \to \infty.$$

Hence the position of the saddle point is determined by

$$t_0 = \ln x + \ln \ln x + \frac{\ln \ln x}{\ln x} - \tfrac{1}{2}\left(\frac{\ln \ln x}{\ln x}\right)^2 + O\left(\frac{\ln \ln x}{\ln^2 x}\right), \; x \to \infty.$$

According to [4] the path of steepest descent is given by

$$s = s_0 + i\tau, \qquad -\infty < \tau < \infty,$$

where $s_0 = -t_0$. If we are satisfied with the first few terms of the asymptotic expansion it is of course permissible to use the slightly displaced line

$$s = s_1 + i\tau, \qquad -\infty < \tau < \infty,$$

passing through the point

$$s_1 = -\ln x - \ln \ln x - \frac{\ln \ln x}{\ln x} + \tfrac{1}{2}\left(\frac{\ln \ln x}{\ln x}\right)^2,$$

as the path of integration L, see [5].

In the neighbourhood of $s = s_1$ the approximation

$$\Phi(s, x) \approx \Phi(s_1, x) - \frac{\tau^2}{2} \frac{\partial^2 \Phi(s, x)}{\partial s^2}\bigg|_{s=s_1} \tag{2.12}$$

can still be used.

From [6] we get for large negative values of s the following asymptotic behaviour of the exponential integral

$$E_1(s) + \ln s = \frac{e^{-s}}{s}\,(1 + O(s^{-1})).$$

Further we need the following asymptotic relations

$$\Phi(s_1, x) \sim -\ln x - \ln\ln x - \frac{\ln\ln x}{\ln x} + \tfrac{1}{2}\left(\frac{\ln\ln x}{\ln x}\right)^2 + 1$$

and

$$\frac{\partial^2 \Phi(s, x)}{\partial s^2}\bigg|_{s=s_1} \sim 1 - \frac{1}{\ln x}$$

Thus in the vicinity of $s = s_1$, $\Phi(s, x)$ is asymptotically equal to the following expression

$$\Phi_1(\tau, x) = -\ln x - \ln\ln x - \frac{\ln\ln x}{\ln x} + \tfrac{1}{2}\left(\frac{\ln\ln x}{\ln x}\right)^2 + 1 - \frac{\tau^2}{2}\left(1 - \frac{1}{\ln x}\right).$$

This expression is substituted in (2.6) which gives

$$H(x) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{x\Phi_1(\tau, x)\}d\tau$$

$$= \frac{\exp\{xf(x)\}}{2\pi(x \ln x)^x} \int_{-\infty}^{\infty} \exp\{-\frac{x\tau^2}{2}\left(1 - \frac{1}{\ln x}\right)\}d\tau,$$

where

$$f(x) = 1 - \frac{\ln\ln x}{\ln x} + \tfrac{1}{2}\left(\frac{\ln\ln x}{\ln x}\right)^2. \tag{2.13}$$

Finally,

$$H(x) \sim \frac{\exp\{xf(x)\}}{\sqrt{2\pi x}(x \ln x)^x}\left(1 + \frac{1}{2 \ln x}\right), \tag{2.14}$$

which is in very good agreement with the numerical calculations of VAN DE LUNE and WATTEL.

If we calculate the situation of the saddle point with greater accuracy this will result in the addition of a few small terms to the sum between the brackets and in the addition to f(x) of the further terms from the expansion of

$$\Phi(s_0, x) + \ln x + \ln \ln x =$$

$$= s_0 - \frac{1}{x}\{E_1(s_0) + \ln s_0\} + \ln x + \ln \ln x.$$

3. We consider next the possible contributions from the complex saddle points. The geometry of these points is given in a paper by LAUWERIER [7]. He considers the conformal map $z = we^{-w}$ and shows that the z-plane is mapped upon an infinite number of regions in the w-plane, which are designated in fig. 1 by I, II, III, IV, ... . If $w = u + iv$, then the curves $u^2 + v^2 = r^2e^{2u}$ in the complex w-plane correspond to circles $|z| = r$ in the z-plane. For large x we have $r = \frac{1}{x}e^{-\frac{1}{x}} < e^{-1}$, so that the curve corresponding to the circle with radius $r = \frac{1}{x}e^{-\frac{1}{x}}$ consists of two parts, a small closed curve around the origin and a parabolical branch.

The roots of equation (2.10)

$$we^{-w} = \frac{1}{x}e^{-\frac{1}{x}}$$

are marked by crosses in fig. 1.

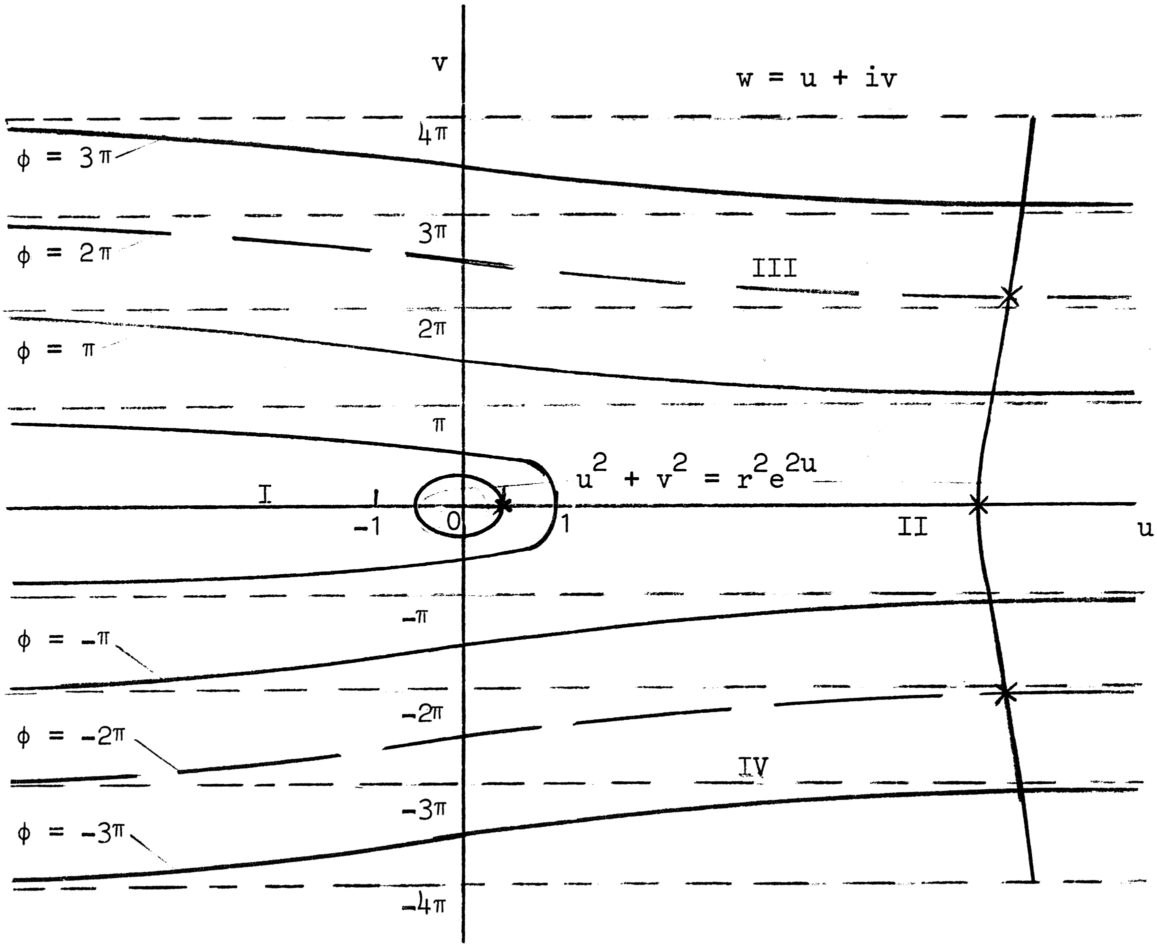


Fig. 1. Conformal mapping

$$z = re^{i\phi} = we^{-w};\ w = u + iv.$$

The s-plane with the path of integration L and the line corresponding to the parabolical line containing the saddle-points in the w-plane are given in fig. 2.

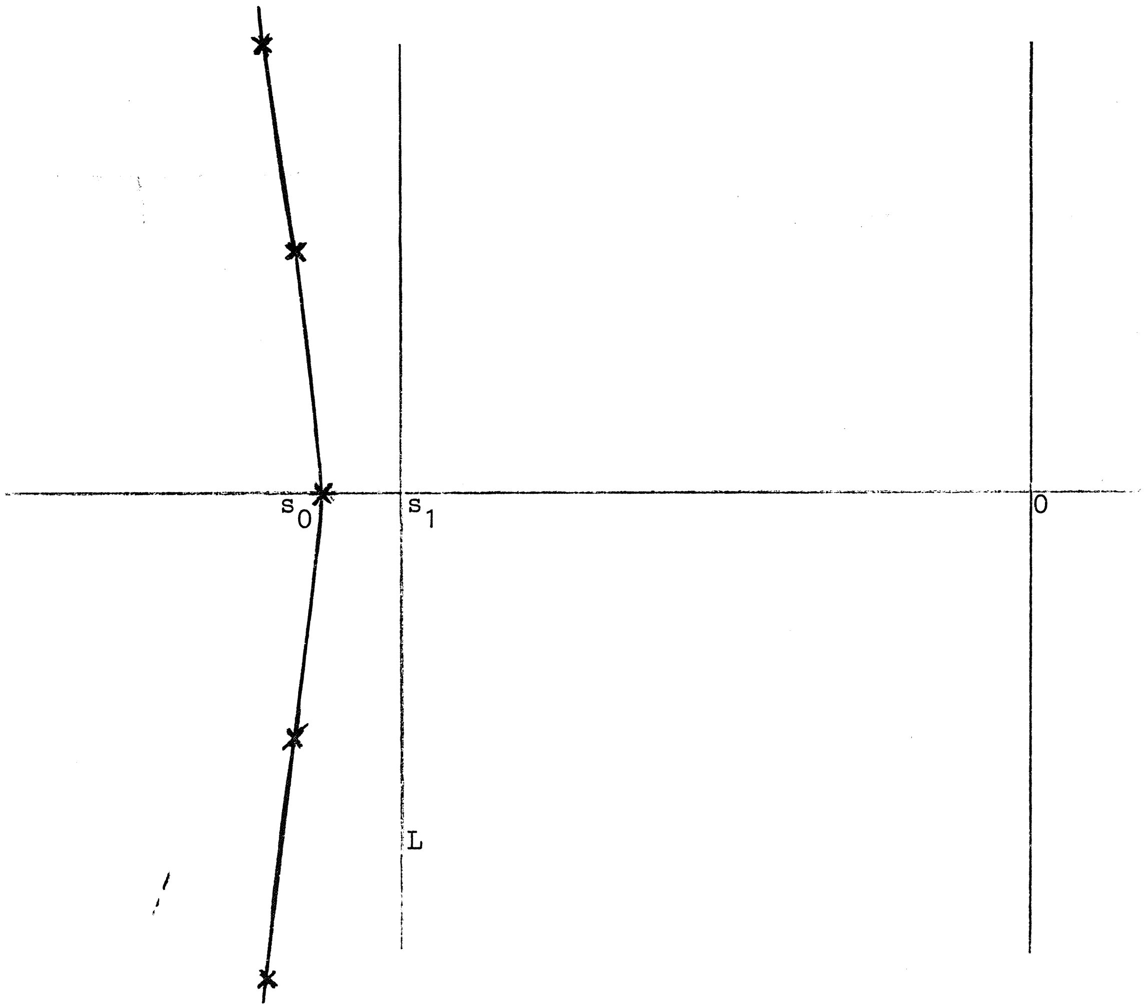


fig. 2.

It is easy to deduce from $u^2 + v^2 = r^2e^{2u}$ that the complex saddle points in the s-plane are all situated on the left-hand side of the vertical through the real saddle point. Therefore the asymptotic behaviour of H(x) is determined only by the real saddle point. Since, however, the complex saddle points are rather near to the vertical line through $s_0$ their contributions to the remainder term can still be considerable. These contributions are studied in great detail by BEENAKKER [2] who also gives explicit estimates.

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